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A Generalized Pólya Algorithm

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It is shown that the convergence of several standard algorithms for the construction of a best approximation to a continuous function on an interval can be established by applying a single theorem which extends the result of Pólya from convergence of sequences of best L_p approximations to that of best approximations with respect to any pointwise convergent sequence of norms. In particular, the convergence of both the first and second Rémès algorithms is obtained as an application of the theorem.

There are several algorithms used to construct the best polynomial approximation to a continuous function f on an interval $[a, b]$. It is shown here that the convergence of these algorithms can be established by applying a single theorem which extends the result of Pólya from convergence of subsequences of best L_p approximations to that of best approximations with respect to any pointwise convergent sequence of norms. Although for clarity we deal only with approximation to a continuous function f on $[a, b]$ by polynomials of degree $n - 1$, the convergence proofs given readily extend to many more general situations discussed in the references. In particular, the convergence of the Rémès algorithms may be established by essentially the same techniques when the space of $n - 1$ degree polynomials is replaced by an n-dimensional space of generalized polynomials, or any n-dimensional subspace satisfying some generalized Haar condition.

The theorem, which depends essentially on the principle of equicontinuity (see, e.g., Dunford and Schwartz [4] p. 53), was established by Kripke [5]. We include a proof for completeness and reference.

THEOREM. *Let X be a real or complex linear space, V an n-dimensional subspace. Suppose $\| \cdot \|_k$ $(1 \leqslant k < \infty)$, $\| \cdot \|$, are norms on $V \oplus \{y\}$, where $y \in X \backslash V$, and $\| x \|_k$ converges to $\| x \|$ for all $x \in X$. Let p_k be a best approximation to y from V with respect to $\| \cdot \|_k$, p a best approximation with respect to $\| \cdot \|$. Then the set of cluster points of $\{p_k : 1 \leqslant k < \infty\}$ is nonempty and contained in the set of best approximations with respect to $\| \cdot \|$; furthermore, if p is unique, $\| p_k - p \|$ converges to zero.*

Proof. We first note that, since $\|\cdot\|_k$ and $\|\cdot\|$ are norms on the finite dimensional space $V \oplus \{y\}$, for each k the norm $\|\cdot\|_k$ is a continuous, subadditive, homogeneous functional on this space considered as a Banach space with norm $\|\cdot\|$. Since the pointwise convergence of $\|\cdot\|_k$ to $\|\cdot\|$ gives the pointwise boundedness of $\{\|\cdot\|_k\}$,

$$\|x\|_k \to 0 \quad \text{as} \quad x \to 0, \qquad \text{uniformly in } k,$$

by the principle of equicontinuity. But each $\|\cdot\|_k$ is a norm, so for each y,

$$\|x\|_k \to \|y\| \quad \text{as} \quad x \to y \qquad \text{and} \qquad k \to \infty.$$

Now clearly $\|\cdot\|_k$ converges uniformly to $\|\cdot\|$ on compact subsets, in particular on $\{x: \|x\| = 1\}$, giving a number M for which $\|x\| \leqslant M\|x\|_k$ for all x, k. Hence

$$\|y - p_k\| \leqslant M\|y - p_k\|_k \leqslant M\|y - q\|_k \to M\|y - q\| \quad \text{as} \quad k \to \infty, \quad \text{any} \quad q \in V,$$

and $\{p_k\}$ is compact. Choosing a subsequence $p_{k'}$ converging to some p, we have

$$\|y - p_{k'}\|_{k'} \to \|y - p\| \quad \text{as} \quad k' \to \infty.$$

Combining the two preceding limits gives $\|y - p\| \leqslant \|y - q\|$ for any $q \in V$, so p is a best $\|\cdot\|$ approximation to y. If p is unique, then every convergent subsequence has the same limit, and p_k converges to p.

Note that in the situation being considered the best Chebyshev (uniform) approximation is unique and the final conclusion of the theorem may be read "p_k converges uniformly to p in $[a, b]$."

Application 1 (Cf. Buck [2] and Burov [3])

The Pólya algorithm is the following: For each (integral) k, $2 \leqslant k < \infty$, find the best approximation p_k to f in the L_k norm. Then p_k converges uniformly to p^*, the best Chebyshev approximation to f.

It is a consequence of standard theorems on inequalities that the L_k norms

$$\|g\|_k \uparrow \|g\| \quad \text{as} \quad k \to \infty$$

for every continuous function g. Thus the theorem applies and uniform convergence of p_k to p^* in $[a, b]$ follows immediately.

Application 2 (Cf. Rice [7])

The de la Vallée Poussin algorithm is the following: Choose a countable dense subset $\{x_j\}$ of $[a, b]$. Find the best Chebyshev approximation p_k to f on

$X_k = \{x_1, ..., x_{n+k}\}$. Then p_k converges uniformly to p^*, the best Chebyshev approximation to f on $[a, b]$.

In order to apply the theorem, we introduce the norms

$$\| g \|_k = \max_{X_k} | g(x)|.$$

Clearly, $\| g \|_k$ converges to $\| g \|_\infty$ for all continuous functions g, and the uniform convergence of p_k to p^* in $[a, b]$ is established.

Application 3 (Cf. Akilov and Rubinov [1])

The first Rémès algorithm is the following: Choose $n + 1$ points $X_1 = \{x_1, ..., x_{n+1}\}$ from $[a, b]$ on which no $n - 1$ degree polynomial interpolates f. Find the best Chebyshev approximation p_1 to f on X_1. Choose a point x_{n+2} where the maximum of $| f - p_1 |$ is attained. Proceed as before, using $X_2 = \{x_1, ..., x_{n+2}\}$. Then p_k converges uniformly to p^*, the best Chebyshev approximation to f on $[a, b]$.

As before, we introduce the norms

$$\| g \|_k = \max_{X_k} | g(x)|, \qquad \| g \| = \max_k \| g \|_k .$$

Applying the theorem, p_k converges uniformly to p, the best $\| \cdot \|$ approximation to f. To show that p is also the best Chebyshev approximation to f on the entire interval, note that since $\{\| \cdot \|_k\}$ is an equicontinuous family

$$\| f - p_{k-1} \|_k \to \| f - p \| \quad \text{as} \quad k \to \infty.$$

But by the choice of X_k

$$\| f - p_{k-1} \|_k = \| f - p_{k-1} \|_\infty \to \| f - p \|_\infty \quad \text{as} \quad k \to \infty.$$

Clearly

$$\| f - p \|_\infty = \| f - p \| \leqslant \| f - q \| \leqslant \| f - q \|_\infty$$

for any $n - 1$ degree polynomial q. Therefore $p = p^*$, which establishes the uniform convergence of p_k to p^* in $[a, b]$.

Application 4 (Cf. Laurent [6])

The second Rémès algorithm is the following: Choose $n + 1$ points $X_1 = \{x_1, ..., x_{n+1}\}$ of $[a, b]$ so that $x_1 < \cdots < x_{n+1}$ and no $n - 1$ degree polynomial interpolates f on X_1. Find the best Chebyshev approximation p_1 to f on X_1. By the classical characterization theorem for Chebyshev approximations, the sign of $f - p_1$ alternates on X_1. Choose points $X_2 = \{x_1', ..., x_{n+1}'\}$ of $[a, b]$, $x_1' < \cdots < x_{n-1}'$, so that the sign of $f - p_1$ alternates on X_2, the minimum of $|f - p_1|$ on X_2 is no less than the maximum of $|f - p_1|$ on X_1 and $|f - p_1|$ attains its maximum on X_2. Now proceed as before, using X_2. In this way a sequence $\{p_k\}$ is generated, where p_k is the best Chebyshev approximation to f on X_k. Then p_k converges uniformly to the best Chebyshev approximation to f on $[a, b]$.

We again introduce the norms

$$\| g \|_k = \max_{X_k} | g(x_k)| \qquad 1 \leqslant k < \infty.$$

Note that

$$\| f - p_{k-1} \|_{k-1} \leqslant \min_{X_k} |(f - p_{k-1})(x)| \leqslant \| f - p_k \|_k .$$

The first inequality was a condition used in choosing X_k. The last inequality follows immediately by recalling that both $f - p_{k-1}$ and $f - p_k$ alternate in sign on the $n + 1$ points X_k, so if $|f - p_{k-1}|$ always exceeded $|f - p_k|$ on X_k the nonzero $n - 1$ degree polynomali $p_k - p_{k-1}$ would have n zeros. Now in order to construct a limit norm, associated with each set $X_k = \{x_1, ..., x_{n+1}\}$ we define the vector $\mathbf{x}_k = (x_1, ..., x_{n+1})$. Since these vectors are all elements of the compact set $[a, b]^{n+1}$, there is a subsequence $\mathbf{x}_{k'}$, converging to $\mathbf{y} = (y_1, ..., y_{n+1})$. Clearly $a \leqslant y_1 \leqslant \cdots \leqslant y_{n+1} \leqslant b$ and $\| g \| = \max\{| g(y)|: y \in Y\}$ is a norm exactly when the elements of $Y = \{y_1, ..., y_{n+1}\}$ are distinct. To show that indeed $y_{j-1} \neq y_j$ for all j, suppose that Y has at most n distinct elements. Then there is an $n - 1$ degree polynomial q which interpolates f on Y. Clearly,

$$0 < \| f - p_1 \|_1 \leqslant \| f - p_{k'} \|_{k'} \leqslant \| f - q \|_{k'} \to \| f - q \| = 0 \quad \text{as} \quad k' \to \infty.$$

Therefore the assumption that Y contained fewer than $n + 1$ distinct points was false, and $\| \cdot \|$ is a norm.

To establish the uniform convergence of $\{p_k\}$ in $[a, b]$, we will again exploit the fact that $|f - p_{k-1}|$ attains its maximum on X_k. To do this choose a subsequence so that $\mathbf{x}_{k'}$ converges to $\mathbf{y}$ and $\mathbf{x}_{k'-1}$ converges to $\mathbf{z} = (z_1, ..., z_{n-1})$, and introduce the norms

$$\| g \| = \max_Z | g(z)|, \qquad \| g \|^* = \max_Y | g(y)|$$

with $Z = \{z_1, ..., z_{n+1}\}$. Applying the theorem, $p_{k'-1}$ converges uniformly to p in $[a, b]$, where p is the best $\| \cdot \|$ approximation to f. It remains only to show that p is the best Chebyshev approximation to f on the entire interval. Since

$\|f - p_k\|_k$ increases, $\|f - p\| = \|f - p^*\|^*$, where p^* is the best $\|\cdot\|^*$ approximation. Again using the equicontinuity of the family $\{\|\cdot\|_k\}$,

$$\|f - p_{k'-1}\|_\infty = \|f - p_{k'-1}\|_{k'} \to \|f - p\|^* \quad \text{as} \quad k' \to \infty,$$

and clearly

$$\|f - p\|_\infty = \|f - p\|^* = \|f - p^*\|^*,$$

the last equality following because $p - p^*$ can have at most $n - 1$ zeros when $p \neq p^*$. Thus $f - p$ equi-oscillates on Y, and again using the characterization theorem, p is the best Chebyshev approximation to f on $[a, b]$. Since p is unique, it follows that p_k converges to p uniformly in $[a, b]$.

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